On the theory of hypersonic flow past plane and axially symmetric bluff bodies

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SUMMARY

The 'Newtonian-plus-centrifugal' approximate solution (Busemann (1933) and Ivey (1948)) for hypersonic flow past plane and axially symmetric bluff bodies in gases with the ratio of the specific heats γ constant and equal to unity is rederived using 'boundary layer' techniques together with the von Mises variables x and ψ. A method of successive approximations then gives a closer approximation to this solution for $\epsilon = (\gamma-1)/(\gamma+1)$ small and the free-stream Mach number infinite. Formulae for the streamlines, shock shape and pressure distribution are determined to this approximation. These formulae are valid for any plane or axially symmetric shape, giving the 'stand-off' distance of the shock wave from the body as $\frac{1}{2}\epsilon\log(4/3\epsilon)$ and ϵ times the nose radius of curvature for plane and axially-symmetric flows respectively. Particular results are computed for a number of special shapes. For certain shapes, the theory has a singular point where the first approximation to the pressure vanishes ($\theta = 60°$ for a sphere). Actually, the theory is not applicable where the pressure becomes too small. The corresponding theory for gases of general thermodynamic properties is deduced, the approximation being valid provided the total energy of the gas is large compared with the energy contained in the translational modes of the gas molecules.

1. INTRODUCTION

The term 'hypersonic' has been coined to describe flow regimes where the free-stream Mach number is considerably in excess of unity. Theoretically, several useful approximate solutions have been obtained for uniform hypersonic flow over slender bodies where the bow shock wave is attached and the fluid is assumed to be inviscid (Van Dyke 1954, Lees 1956). For bluff bodies, however, the bow shock wave is detached, and a mixed subsonic–supersonic region exists behind it. This makes the theoretical solution of the problem considerably more difficult. Nevertheless, by making the additional assumption that the ratio of the specific heats γ of the gas is near one, an approximation to the pressure on the surface of a symmetrical bluff body in an inviscid flow can be obtained (Ivey, Klunker & Bowen 1948, Busemann 1933). This is known as the 'Newtonian-plus-centrifugal' solution for reasons which will be seen below.

Behind an oblique shock wave in a uniform flow of velocity U, density ρ_0 and Mach number M, the normal component of velocity u_n is given (§ 2) by

$$u_n = \frac{\gamma-1}{\gamma+1} U \sin\Theta + O\left(\frac{1}{M^2}\right)$$

as $M \to \infty$, where the gas is assumed to be perfect with constant specific heats, and where Θ is the angle the shock makes with the oncoming flow. We see that as $\gamma \to 1$, $u_n \to 0$. Thus, in the limiting case of a strong shock wave and γ near unity, it is possible for the bow shock wave to rest actually on the body, since the boundary condition for an inviscid fluid, that the normal component of velocity is zero, is satisfied. Fluid will still cross the shock, however; for the density behind the shock is

$$\rho = \rho_0 \left\{\frac{\gamma+1}{\gamma-1} + O\left(\frac{1}{M^2}\right)\right\},$$

which becomes infinite as $\gamma \to 1$, and hence the mass flow ρu_n across the shock wave remains finite. This fluid can be squeezed between the shock and the body, since the stream-tube area which is proportional to $\rho_0 U/\rho q$ becomes small for $\gamma \sim 1$ (q is the velocity behind the shock wave and remains finite as $\gamma \to 1$ due to continuity of the tangential velocity $u_t = U\cos\Theta$ across the shock wave). The pressure behind the shock wave is

$$p = \frac{2}{\gamma+1}\rho_0 U^2 \sin^2\Theta \sim \rho_0 U^2 \sin^2\Theta$$

as $\gamma \to 1$. This was given as the pressure on the body surface by Epstein (1931); but, as we shall see, it is incorrect for bodies of finite curvature. The name 'Newtonian' is given to this result, since it is the pressure on a body surface placed in a stream of inelastic particles as given by Newton (1689), the particles losing all their momentum normal to the surface on impact. It would seem that to attempt to imply a closer relationship is questionable, for the highly compressed gas discussed here bears little resemblance to the "*medium rarum quod ex particularis quam minimis quiescentibus aequilibus et ad aequalis ab invicem distantes libere dispositis constat*" which Newton postulates.

For bodies of finite curvature, however, the pressure on the body differs from that upon the shock, since there is a finite amount of fluid in the narrow region between the two. The pressure difference is equal to the centrifugal force on this layer of fluid. Taking this into account leads to the 'Newtonian-plus-centrifugal' solution given by Ivey, Klunker & Bowen (1948) and anticipated by Busemann (1933).

In this paper, we will follow these authors in using the 'strong shock' approximation for large M, but will assume $\epsilon = (\gamma-1)/(\gamma+1)$ to be small but not zero, with the solution for $\epsilon = 0$ as a first approximation in the solution for infinite Mach number. This successive approximation procedure is something like expanding in powers of ϵ; the solution will be obtained as far as the first term in ϵ.

In §2, the conditions across the shock wave are derived from the Rankine–Hugoniot relations. A further simplification is introduced by assuming the gas to be perfect with constant specific heats. In practice, of course, the conditions for such an assumption to be valid will not prevail. However, the theory will be developed throughout using this assumption in order to obtain a definite result. The modifications to be introduced if this is not so are discussed in § 7, where the solution for a gas with arbitrary thermodynamic properties is given.

The 'Newtonian-plus-centrifugal' solution for two- and three-dimensional flow is then obtained in §3 from the equations of momentum, continuity and energy for a compressible inviscid flow without heat conduction. We assume that between the shock wave and the body there is a thin 'boundary layer' in which changes perpendicular to the body surface are large compared with those along the body. As for other boundary layers, we assume the layer thickness is small compared with the body dimensions and that the velocity component normal to the body is small compared with that along the body, or, alternatively, that the streamlines deviate only slightly from the body shape. A measure of the magnitude of the variables being given by their values on the shock wave, the density will be of the order of ρ_0/ϵ and the pressure of the order of $\rho_0 U^2$. We must not, however, be led into making too close a parallel between this boundary layer and the classical viscous boundary layer where in most cases any centrifugal effects are negligible. Introducing the ordinary boundary layer coordinates (x, y), the momentum equation in the x-direction, approximated in the above manner, states that the velocity component u in the x-direction is constant along streamlines. (This is because with ϵ small the enthalpy changes very little in an adiabatic expansion.) The momentum equation in the y-direction gives the pressure gradient normal to the body as due to the centrifugal forces produced by body curvature alone. From this, the pressure at any point of the layer is obtained as a function of the stream function ψ (or, in axially symmetric flow, the Stokes stream function) and the coordinate x. Using the energy equation in the form of constancy of entropy along streamlines, the density can also be obtained in terms of these variables. To this approximation the shock is assumed to have the body shape. Finally, the position of each streamline can be derived by integration from a knowledge of ρ and u as functions of x and ψ. By replacing ψ by its first approximation on the shock wave in this expression, we obtain the shock shape. This result is given in §4 and plotted in figure 3 for the cases of the circular cylinder, parabolic cylinder, sphere and paraboloid of revolution. Knowledge of the streamlines of the flow and the shock shape enables us to calculate the small component of velocity v parallel to y and a higher approximation to u, and hence obtain a second approximation to the pressure (§6). In particular, the variation of the pressure on the surface of a sphere is computed and plotted in figure 5.

The velocity tangential to the shock wave increases as we move away from the stagnation point; and hence, on many bluff bodies (those on which the curvature does not decrease with distance from the stagnation point), the centrifugal force on the fluid will increase with increasing distance from the stagnation point. The pressure rise across the shock wave, however, being proportional to the normal velocity, will decrease. Consequently, the pressure drop between the shock and the body may eventually annul the pressure rise across the shock wave, and the pressure as calculated from the theory will fall to zero on the body. The point where this happens is a singular point of the theory, the approximation not being valid there. This difficulty arises from the failure of the first approximation at this point. It is no longer possible to assume that the stream-tube area is still small enough for the shock wave to rest approximately on the body surface once the pressure on the body has fallen to a small fraction of $\rho_0 U^2$. The problem then arises of finding the first approximation to the shock shape in this region. It would seem impossible to do this without making some further assumption, since, immediately we admit the layer between the shock wave and the body to be no longer thin, we are beset with a further difficulty of having two velocity components of the same order of magnitude. The form which this further assumption should take remains a matter for conjecture. The streamlines as they emerge from the narrow layer presumably fan out to cover the region between the shock and the body. Several assumptions have been tried as to the nature of this fanning-out, such as allowing the gas to separate from the surface, but none seems to give results of the magnitude required from experimental evidence. They will not, therefore, be further discussed. The restrictions implied by this singularity can best be seen by consideration of a few examples. Unfortunately, it would appear that the circular cylinder and sphere are particularly bad from this point of view, the approximation breaking down at $54^\circ 44'$ and 60° respectively from the front stagnation point. On a parabolic cylinder and paraboloid, the position is much more favourable, the singular point being situated at infinity in both cases. However, even on these shapes a position is reached where the pressure falls to a value too small for the theory to be valid.

The assumptions underlying the theory introduced in this paper, namely, that the Mach number is infinite and the ratio of the specific heats of the gas is nearly unity, are only approximately true in practice. In the hypersonic regime the ratio of the specific heats for air reaches a minimum of about 1·2. At large Mach numbers (and hence high temperatures) dissociation of the gas is almost certain to be well advanced. Variation of the molecular weight of the gas with temperature and pressure and, more important, the large variation in specific heats must then be taken into account. This theory can be extended (§ 7) to cover gases of quite arbitrary thermodynamic properties, provided the parameter ϵ is replaced by

$K^{-1} = \rho_0/\rho_s$ (where ρ_s is the density on the shock wave). The parameter K is not necessarily constant, however. The solution obtained in § 7 holds for large values of K.

An interesting feature of the flow discussed here is the apparently large shear in the layer between the shock and the body, even on the assumption of inviscid flow. This shear is almost linear (figure 4). Moreover, the component of velocity u, since it is constant to the first approximation along streamlines, is zero upon the body itself (as the body surface is the streamline through the stagnation point). This being so, the non-slip condition at the surface for a viscous fluid is already satisfied to this approximation. It would seem that, on this account, viscosity may not play such an important part in the flow as expected. The theory does, however, neglect heat conduction, which will be especially important in regions near the stagnation point where gas temperatures are large. Nevertheless, this may not have too disastrous an effect, as we may expect heat conduction to lead to lower temperatures at the body surface and hence a thinning of the boundary layer.

A further difficulty which arises in considering a real gas is that the equilibrium assumed behind the shock wave may take some time to achieve. Relaxation times may be quite large for vibration and dissociation, which would be excited at the high temperatures the gas is likely to experience. This would cause the ratio of the specific heats to vary considerably in the flow immediately behind the shock wave. The relaxation time being roughly proportional to the density, the importance of this effect is determined by the size of the parameter $(\rho d)^{-1}$, where d is a length determining the scale of the flow.

The results given here seem to agree fairly well with a solution given by Lighthill (1956) in cases where K is large. This solution refers to conditions near the stagnation point of a sphere, comparison being made only in that region. The 'stand-off' distance ϵ of the shock wave on a sphere disagrees, however, with the value $\frac{1}{2}\epsilon$ given by Schwartz & Eckermann (1955). This may be due to their making some assumptions with regard to the flow near the stagnation point. Their experimental results would seem to confirm neither value. Experimental results given by Oliver (1956) (on flow past a cylinder with a hemispherical nose) for the pressure on the body surface seem to agree more closely with the original 'Newtonian' solution than the modified one. The Mach number in these experiments was, however, still quite low (about six).

During the preparation of this paper for publication, the author's attention was drawn to the (then unpublished) work by Chester (1956). Chester also treats the present problem by a process of successive approximation, starting with the 'Newtonian-plus-centrifugal' solution. It was decided, however, to publish the two results separately as his approach to the problem is different from the present one, and also the further development of the theory proceeds along different lines. In so far as it is possible to compare Chester's results with those given here, they are identical.

2. The shock conditions

An oblique shock wave makes an angle Θ with a uniform flow of velocity U, the density, pressure and enthalpy in the flow being ρ_0, p_0 and i_0 respectively (figure 1). Behind the shock the velocities tangential and normal to the shock wave are u_t and u_n; and the pressure, density and enthalpy are p_1, ρ_1 and i_1 respectively. Normal to the shock wave the Rankine–Hugoniot equations (stating continuity of mass flow, momentum

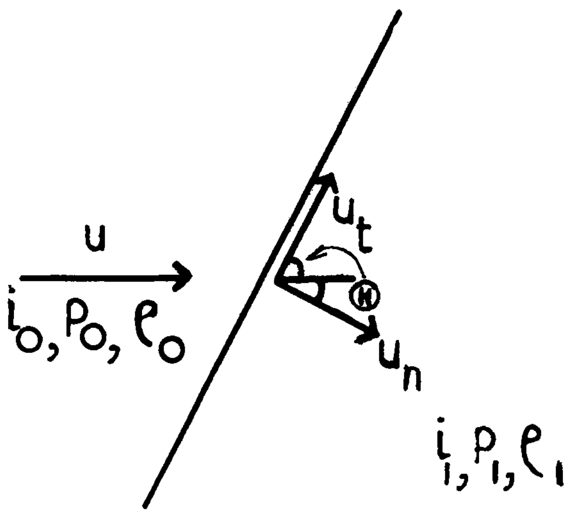

Figure 1. The variables at the shock wave.

and energy) hold. Tangential to the shock wave the component of velocity is continuous. From these relations, together with a knowledge of the thermodynamics of the gas, we obtain the conditions behind the shock wave. Thus,

$$\left.\begin{aligned} \rho_0 U \sin\Theta &= \rho_1 u_n, \\ p_0 + \rho_0 U^2 \sin^2\Theta &= p_1 + \rho_1 u_n^2, \\ i_0 + \tfrac{1}{2} U^2 \sin^2\Theta &= i_1 + \tfrac{1}{2} u_n^2, \\ U\cos\Theta &= u_t, \end{aligned}\right\} \tag{2.1}$$

and

$$i_1 = i_1(p_1, \rho_1), \tag{2.2}$$

the latter equation being a purely thermodynamical relationship. For hypersonic flow the dynamic terms on the left-hand side are much larger than the state variables p_0 and i_0. Thus, the equations (2.1) may be approximated as

$$\left.\begin{aligned} \rho_0 U \sin\Theta &= \rho_1 u_n, \\ \rho_0 U^2 \sin^2\Theta &= p_1 + \rho_1 u_n^2, \\ \tfrac{1}{2} U^2 \sin^2\Theta &= i_1 + \tfrac{1}{2} u_n^2, \\ U\cos\Theta &= u_t \end{aligned}\right\} \tag{2 3}$$

Alternatively, equation (2.3) can be written in terms of $K = \rho_1/\rho_0$ as

$$\frac{u_n}{U\sin\Theta} = \frac{1}{K}, \quad \frac{p_1}{\rho_0 U^2 \sin^2\Theta} = 1-\frac{1}{K}, \quad \frac{i_1}{U^2\sin^2\Theta} = \frac{1}{2}\left(1-\frac{1}{K^2}\right), \tag{2.4}$$

and

$$u_t = U\cos\Theta.$$

Also,

$$\frac{i_1\rho_1}{p_1} = \frac{1+K}{2}. \tag{2.5}$$

In general, the shock equations (2.4) are solved for K by substituting i_1 and p_1 into the thermodynamic relation in the form $\rho = \Omega(p,i)$. If K is large, however, a method of successive approximation is found to converge very rapidly; for i_1 is known from (2.4) to be very nearly $\frac{1}{2}U^2\sin^2\Theta$, and for given i_1 the ratio $i_1\rho_1/p_1$ (which by (2.5) determines K) varies only slowly with ρ_1 or with p_1 and so can be determined by trying successively different values of ρ_1 and p_1. This ratio is 3/2 times the ratio of the total enthalpy to the translational energy of the molecules, which would be expected to be large for real gases in hypersonic flow. Hence, the assumption that K is large can be interpreted as $i\rho/p$ being large in the region behind the shock wave.

For K large, we see from equation (2.4) that the normal component of velocity is small, and the density large behind the shock wave.

For a perfect gas with constant specific heats, K is a constant independent of Θ, and equals $\epsilon^{-1} = (\gamma+1)/(\gamma-1)$. Thus the assumption of large K requires γ to be nearly unity for such a gas. The theory will be developed for this ideal gas in the following sections.

3. The equations of motion

Consider flow past a symmetrical bluff body either plane or axially symmetric, placed in a uniform stream of velocity U. The shock wave

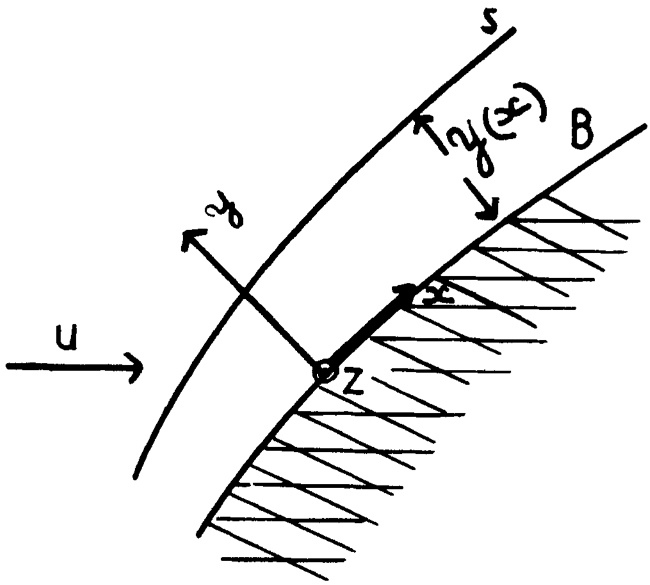

Figure 2. The curvilinear coordinate system used in the boundary layer. S, shock; B, body.

in front of the body will be detached and lie a distance δ—usually referred to as the 'stand-off' distance—upstream of the stagnation point. We introduce the usual boundary layer coordinates (x, y) (figure 2). The y-axis is normal to the body surface, and the x-axis lies along the body in the plane formed by the normal and the direction of the uniform stream. A further coordinate z is introduced on the body surface, such that (x, y, z) form a right-handed system of axes. The elements of length in the (x, y, z) directions are denoted by $h\,dx$, dy and $k\,dz$ respectively. The case of two-dimensional flow can be reproduced by putting $k = 1$.

The equations of momentum, continuity and energy may then be written

$$\left.\begin{aligned} \frac{u}{h}\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+uv\kappa+\frac{1}{h\rho}\frac{\partial p}{\partial x} &= 0,\\ \frac{u}{h}\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-u^2\kappa+\frac{1}{\rho}\frac{\partial p}{\partial y} &= 0,\\ \frac{\partial}{\partial x}(k\rho u)+\frac{\partial}{\partial y}(hk\rho v) &= 0,\\ \frac{u}{h}\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y} &= 0, \end{aligned}\right\} \tag{3.1}$$

where S is the entropy and κ the curvature of the line y = constant. We assume now that the stream tube area is small behind the shock wave when ϵ is small, and it is therefore possible, in the limit $\epsilon \to 0$, for the shock to rest on the body. For small ϵ we assume *a priori* that the distance of the shock from the body is $O(\epsilon)$, and that slope of the shock relative to the body is $O(\epsilon)$. (In the two-dimensional case it will be seen that it is later necessary to modify this assumption. These magnitudes then become $O(\epsilon\log\epsilon)$.) On the shock, $u_n/U = O(\epsilon)$ and $u_t/U = O(1)$; and thus, resolving in the direction of the coordinate axes, we have $u/U = O(1)$ and $v/U = O(\epsilon)$ on the shock, also $p/\rho_0 U^2 = O(1)$ and $\rho/\rho_0 = O(\epsilon^{-1})$ from (2.4). In the flow behind the shock wave, these variables will have the same orders of magnitude. Changes across the thin boundary layer between the shock and the body will be large compared with those along it. Assuming that the above conditions hold, except perhaps in regions near the stagnation point, we introduce the new variables

$$p' = \frac{p}{\rho_0 U^2}, \qquad \rho' = \frac{\epsilon\rho}{\rho_0}, \qquad u' = \frac{u}{U}, \qquad v' = \frac{v}{U\epsilon}, \tag{3.2}$$

and

$$x' = \frac{x}{d}, \qquad y' = \frac{y}{\epsilon d};$$

and then the dashed variables are all $O(1)$. Substituting in (3.1) and noting that h, k and κ are $O(1)$, we obtain

$$\left.\begin{aligned} \frac{u'}{h}\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'} &= O(\epsilon),\\ \frac{1}{\rho'}\frac{\partial p'}{\partial y'}-u'^2\kappa &= O(\epsilon),\\ \frac{\partial}{\partial x'}(k\rho' u')+\frac{\partial}{\partial y'}(hk\rho' v') &= 0. \end{aligned}\right\} \tag{3.3}$$

Assuming the gas to be perfect, we have $S = p/\rho^{\gamma}$ and hence

$$\frac{u'}{h}\frac{\partial}{\partial x'}\left(\frac{p'}{\rho'}\right)+v'\frac{\partial}{\partial y'}\left(\frac{p'}{\rho'}\right) = O(\epsilon \log \epsilon). \tag{3.4}$$

Neglecting terms of higher order than the first, we therefore have, in terms of the original variables,

$$\left.\begin{aligned} \frac{u}{h}\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= 0,\\ \frac{1}{\rho}\frac{\partial p}{\partial y} &= u^2\kappa,\\ \frac{u}{h}\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)+v\frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) &= 0, \end{aligned}\right\} \tag{3.5}$$

with the continuity equation remaining unchanged. The first equation of (3.5) states that the velocity component u is constant along streamlines, and the final equation states that the enthalpy is constant along streamlines. This is consistent with Bernoulli's equation which is, to this approximation,

$$\tfrac{1}{2}u^2+\left(\frac{1+\epsilon}{2\epsilon}\right)\frac{p}{\rho} = \tfrac{1}{2}U^2. \tag{3.6}$$

The second equation of (3.5) demands that the pressure gradient normal to the body shall be that due to the centrifugal force produced by the curvature of the body alone.

Introducing a stream function ψ to satisfy the continuity equation, we require that

$$k\rho u = \frac{\partial\psi}{\partial y}, \qquad hk\rho v = -\frac{\partial\psi}{\partial x}. \tag{3.7}$$

Equations (3.5) then state that u and p/ρ are functions of ψ alone, and that

$$\frac{\partial p}{\partial \psi} = \frac{u\kappa}{k}. \tag{3.8}$$

4. The equation of the shock and streamlines in two and three dimensions

The mass of fluid flowing across the shock wave is ρu_n, where u_n is the velocity normal to the shock. Now,

$$\rho u_n = \rho(ul+vm), \tag{4.1}$$

where l, m are the direction cosines of the normal to the shock wave. By using (3.7), this may be written

$$\rho u_n = \frac{1}{k}\left(\frac{l_1}{h}\frac{\partial\psi}{\partial x}+m_1\frac{\partial\psi}{\partial y}\right), \tag{4.2}$$

where l_1, m_1 are the direction cosines of the tangent to the shock wave. If s is the distance measured along the shock wave, therefore,

$$\rho u_n = \frac{1}{k}\frac{d\psi}{ds}. \tag{4.3}$$

On the shock wave $\rho u_n = \rho_0 U \sin\Theta$, where Θ is the angle of the shock to the oncoming flow, or, in terms of s,

$$\rho u_n = \rho_0 U \frac{d\eta_s}{ds}, \tag{4.4}$$

where $\eta_s(x)$ is the distance of the shock from the line of symmetry. Hence, from (4.3),

$$\rho_0 U \frac{d\eta_s}{ds} = \frac{1}{k}\frac{d\psi}{ds}, \tag{4.5}$$

or

$$\psi = \rho_0 U \int k\, d\eta_s. \tag{4.6}$$

Let us now assume $\psi = 0$ on the line of symmetry; and then, since $k = 1$ in two dimensions and $k = \eta_s$ in three dimensions, we have

$$\left.\begin{aligned} \psi &= \rho_0 U \eta_s, \\ &= \tfrac{1}{2}\rho_0 U \eta_s^2, \end{aligned}\right\} \tag{4.7}$$

in two and three dimensions respectively. If $\eta(x)$ is the distance of a point on the body itself away from the line of symmetry, then

$$\eta_s = \eta(x) + \mathcal{Y}(x)\cos\Phi(x), \tag{4.8}$$

where Φ is the angle of the body to the uniform flow direction and $y = \mathcal{Y}(x)$ is the position of the shock. Thus, if $\mathcal{Y}(x)$ is known, equation (4.7) together with equation (4.5) give ψ as a function of x on the shock wave. On the assumption that $\mathcal{Y}(x)$ is small, however,

$$\begin{aligned} \psi &= \rho_0 U \eta(x) \\ &= \tfrac{1}{2}\rho_0 U \eta^2(x) \end{aligned} \tag{4.9}$$

in two and three dimensions respectively, if we neglect $O(\epsilon)$. Thus we know ψ explicitly on the shock wave: $\psi = \rho_0 U \Psi(x)$, say.

Also, $$\Theta = \Phi + \tan^{-1}\{h^{-1}(d\mathcal{Y}/dx)\},$$

which gives on neglecting $O(\epsilon)$,

$$\Theta = \Phi. \tag{4.10}$$

Thus the shock conditions (§ 2) reduce to

$$\left.\begin{aligned} u &= U[\cos\Phi + O(\epsilon)], \\ v &= O(U\epsilon), \\ \frac{\rho}{\rho_0} &= \frac{1}{\epsilon}, \\ \frac{p}{\rho_0 U^2} &= \sin^2\Phi + O(\epsilon), \end{aligned}\right\} \tag{4.11}$$

which together with (4.9) give the dependent variables as functions of ψ on the shock wave.

If we now introduce a new variable ξ defined by $\psi = \rho_0 U \Psi(\xi)$, then we can replace ψ by ξ which is the x coordinate of the point where the streamline ψ crosses the shock wave.

From the equations of motion, u and p/ρ are functions of ψ, and hence of ξ, alone. Thus, using (4.11) we obtain

$$u = U\cos\Phi(\xi), \tag{4.12}$$

and

$$\frac{p}{\rho} = \epsilon U^2 \sin^2\Phi(\xi)$$

throughout the flow field.

Also, from (3.8)

$$\left.\begin{aligned} p &= \int \frac{u\kappa}{k}\,d\psi \\ \text{or}\qquad p &= p_s(x) + \int_{\psi_s}^{\psi} \frac{u\kappa}{k}\,d\psi, \end{aligned}\right\} \tag{4.13}$$

where the suffix s refers to conditions on the shock wave at a station x. Now, $\kappa = \kappa_{\psi=0} + O(\epsilon)$ and $k = k_{\psi=0} + O(\epsilon)$; and hence

$$\left.\begin{aligned} p &= p_s + \frac{\kappa_0}{k_0}\int_{\psi s}^{\psi} u\,d\psi \\ &= p_s + \rho_0 U \frac{\kappa_0}{k_0}\int_{x}^{\xi} u\Psi'(t)\,dt, \end{aligned}\right\} \tag{4.14}$$

where $u = u(x, \psi = \rho_0 U\Psi(t))$ and $\kappa_{\psi=0} = \kappa_0$, etc. From (4.11) and (4.12) we obtain

$$p_s = \rho_0 U^2 \sin^2\Phi(x) \quad \text{and} \quad u = U\cos\Phi(\xi);$$

and therefore

$$\frac{p}{\rho_0 U^2} = \sin^2\Phi(x) + \frac{\kappa_0}{k_0}\int_x^{\xi} \Psi'(t)\cos\Phi(t)\,dt. \tag{4.15}$$

This is the result given by Busemann (1933) and Ivey (1948). Again using (4.12), we obtain

$$\frac{\rho}{\rho_0} = \frac{1}{\epsilon}\left[\frac{\sin^2\Phi(x) + \dfrac{\kappa_0}{k_0}\displaystyle\int_x^{\xi} \Psi'(t)\cos\Phi(t)\,dt}{\sin^2\Phi(\xi)}\right]. \tag{4.16}$$

It is now possible to reintroduce the coordinate y into the problem. For, from (3.7), we have

$$y = \int_0^{\psi} \frac{1}{k\rho u}\,d\psi, \tag{4.17}$$

keeping x constant; or, making the previous approximations and introducing (4.16), we obtain

$$y = \frac{\epsilon}{k_0}\int_0^{\xi} \frac{\Psi'(t)\sin^2\Phi(t)\,dt}{\left\{\sin^2\Phi(x) + \dfrac{\kappa_0}{k_0}\displaystyle\int_x^{\xi}\Psi'(s)\cos\Phi(s)\,dS\right\}\cos\Phi(t)}$$

for the equation of the streamlines.

The above integral does not converge for the two-dimensional case, however, since $\cos\Phi(t) = O(t)$ as $t \to 0$. In the three-dimensional case the integrand has no such singularity, since the function $\Psi' = O(t)$ also.

In the two-dimensional problem, therefore, it is necessary to obtain a better approximation to u which will be valid for ψ or ξ small. It will be seen that magnitudes of the order of $\epsilon \log \epsilon$ will replace those taken to be $O(\epsilon)$ in the previous approximation; and thus the error terms previously denoted by $O(\epsilon)$ must be replaced by $O(\epsilon \log \epsilon)$. Bernoulli's equation in the form (3.6) may still be expected to hold near the body, since $v = 0$ by the boundary condition there. The energy equation (3.1) for a perfect gas with constant specific heats may be written

$$\frac{p}{\rho^\gamma} = \frac{p_*}{\rho_*^\gamma}, \tag{4.18}$$

where p_* and ρ_* are the values of p and ρ where the streamline crosses the shock wave. Equations (4.18) and (3.6) together then give

$$\tfrac{1}{2}u^2 + \left(\frac{1+\epsilon}{2}\right)\frac{p_*}{\rho_0}\left\{1 + 2\epsilon \log \frac{p}{p_*}\right\} = \tfrac{1}{2}U^2[1 + O(\epsilon^2 \log^2 \epsilon)]. \tag{4.19}$$

Also, the pressure on the shock wave is, from (2.4),

$$p = (1-\epsilon)\rho_0 U^2 \sin^2\left\{\Phi + \tan^{-1}\frac{1}{h}\frac{d\mathscr{Y}}{dx}\right\}. \tag{4.20}$$

Substituting in (4.19), we obtain

$$u^2 = U^2\left[\cos^2\Phi(\xi) - \sin^2\Phi(\xi)\left\{2\epsilon \log\left(\frac{p(x,\xi)}{p_*(\xi)}\right) + 2\mathscr{Y}'(\xi)\cot\Phi(\xi)\right\} + {} \right.$$
$$\left. {} + O(\epsilon^2 \log^2 \epsilon)\right], \tag{4.21}$$

where $p_* = p(\xi, \xi)$ and p is given by (4.15).

The term in curly brackets is $O(\epsilon \log \epsilon)$, and becomes important near the body surface where the first term is small. Since, however, this higher approximation is important only near the body surface, it will be sufficient to substitute its value actually on the body to obtain the position of the streamlines from (4.17). Thus, putting $\psi = \xi = 0$ in the second term of (4.21), the required approximation for the velocity component u is

$$u = U\left[\cos^2\Phi(\xi) - 2\epsilon \log \frac{p(x,0)}{\rho_0 U^2}\right]^{1/2}, \tag{4.22}$$

where $p(x, \xi)$ is given by (4.15) (since $\mathscr{Y}'(\xi) \sim 0$ and $p(\xi, \xi) \sim \rho_0 U^2$ as $\xi \to 0$). In the two-dimensional case this will replace the original value of u in (4.17). Thus the equation for the streamlines $\xi =$ constant is given by

$$y = \epsilon \int_0^\xi \frac{\sin^3\Phi(t)\, dt}{R(x,t)\{\cos^2\Phi(t) - 2\epsilon \log R(x,0)\}^{1/2}}, \tag{4.23}$$

where
$$R(x,t) = \sin^2\Phi(x) + \Phi' \int_t^x \cos\Phi(s) \sin\Phi(s)\, dS$$

in two dimensions, and

$$y = \epsilon \int_0^\xi \frac{\eta(t)\sin^3\Phi(t)\, dt}{Q(x,t)\cos\Phi(t)}, \tag{4.24}$$

where
$$Q(x,t) = \sin^2\Phi(x) + \frac{\Phi'}{\eta}\int_t^x \cos\Phi(s)\sin\Phi(s)\eta(s)\, ds$$

in three dimensions, the variables k_0, κ_0 and η' having been replaced by $\eta(x)$, $-\Phi'$ and $\sin\Phi$ respectively. On the shock wave, $\xi = x$, and hence the equation of the shock wave is obtained by replacing ξ by x in (4.23) or (4.24).

The validity of three formulae near the stagnation point is doubtful since the approximation that one velocity component is much larger than the other may not be true there, for *both* components of velocity are small in the neighbourhood. It will be shown later (§ 5), however, that the formulae remain valid. Using (4.23) and (4.24), therefore, we can obtain the 'stand-off' distances in two and three dimensions as

$$\frac{\delta}{a} = \frac{1}{2}\epsilon\log\frac{4}{3\epsilon}, \tag{4.25}$$

and

$$\frac{\delta}{a} = \epsilon \tag{4.26}$$

by noting that $\eta(t) \sim t$ and $\Phi \sim \frac{1}{2}\pi - t/a$ for t small, where a is the radius of curvature of the nose of the body.

Geometrically, the simplest types of bluff bodies are, of course, the circular cylinder and the sphere. For these, $\Phi = \frac{1}{2}\pi - \theta$ and $\eta(x) = a\sin\theta$, where $\theta(= x/a)$ is the angle in the plane (or meridian plane) measured from the front stagnation and a is the radius. The equations (4.23) and (4.24) then give

$$\begin{aligned}\frac{r-a}{a} &= 2\epsilon\int_0^{\xi/a}\frac{\cos^3 t\,dt}{[3\cos^2\theta-\cos^2 t]\{\sin^2 t - 2\epsilon\log(1-\frac{3}{2}\sin^2\theta)\}^{1/2}}\\ &= \frac{\epsilon}{(3\cos^2\theta-1)}\left\{\log\left(\frac{2\sin^2(\xi/a)}{\epsilon|\log(1-\frac{3}{2}\sin^2\theta)|}\right)-3\cos^2\theta\log\left(1+\frac{\sin^2(\xi/a)}{2-3\sin^2\theta}\right)\right\},\end{aligned} \tag{4.27}$$

to this approximation for a circular cylinder, and

$$\begin{aligned}\frac{r-a}{a} &= \epsilon\int_0^{\xi/a}\frac{\cos^3 t\,dt}{\frac{1}{3}(\sin 3\theta+\sin^3 t)}\\ &= \frac{\epsilon}{\alpha^2}\left[(1-\alpha^2)\log\left(\frac{\alpha+\beta}{\alpha}\right)-\left(\frac{2\alpha^2+1}{2}\right)\log\left(1-\frac{\beta}{\alpha}+\frac{\beta^2}{\alpha^2}\right)+\right.\\ &\qquad\left.+\sqrt{3}\tan^{-1}\frac{\beta\sqrt{3}}{2\alpha-\beta}\right]\end{aligned} \tag{4.28}$$

for a sphere, where $\alpha = \sqrt[3]{(\sin 3\theta)}$, $\beta = \sin(\xi/a)$ and r is a polar coordinate measured from the centre of the sphere or cylinder. The shock shape is deduced by putting $\xi/a = \theta$ in (4.27) or (4.28).

These formulae become useless near the singular point of the theory, given by $R(a\theta, 0) = 0$ for (4.23) and $Q(a\theta, 0) = 0$ for (4.24). For a circular cylinder this is $\theta = 54°\,44'$, and for a sphere $\theta = 60°$. For a parabolic cylinder or paraboloid of revolution, however, this singular point is removed to infinity. Equations (4.23) and (4.24) then become

$$\begin{aligned}\frac{y}{2b} &= \epsilon\left(1+\frac{Y^2}{4b^2}\right)^{3/2}\int_0^{\Xi/2b}\frac{dt}{(1+t^2)^{3/2}[t^2(1+t^2)^{-1}+3\epsilon\log(1+Y^2/4b^2)]^{1/2}}\\ &= \frac{\epsilon}{2}\left(1+\frac{Y^2}{4b^2}\right)^{3/2}\log\left(\frac{\Xi^2}{3\epsilon b^2[1+(\Xi^2/4b^2)\log[1+(Y^2/4b^2)]]}\right)\end{aligned} \tag{4.29}$$

for a parabola, and

$$\frac{y}{2b} = 2\epsilon\left(1+\frac{Y^2}{4b^2}\right)^{3/2}\int_0^{\Xi/2b}\frac{dt}{(1+t^2)^{1/2}[G_+(Y/2b)+G_-(t)]} \tag{4.30}$$

for a paraboloid of revolution, where $G_\pm(t) = t(1+t^2)^{1/2} \pm \sinh^{-1} t$, (X, Y) is a system of Cartesian coordinates (with X along the line of symmetry and Y perpendicular to it) in which the parabola has an equation $Y^2 = 4bX$, and Ξ is the value of Y at the point where the streamline ψ crosses the shock wave. Thus the shock wave is obtained by putting $\Xi = Y$ in (4.29) or (4.30).

These results are plotted in figure 3.

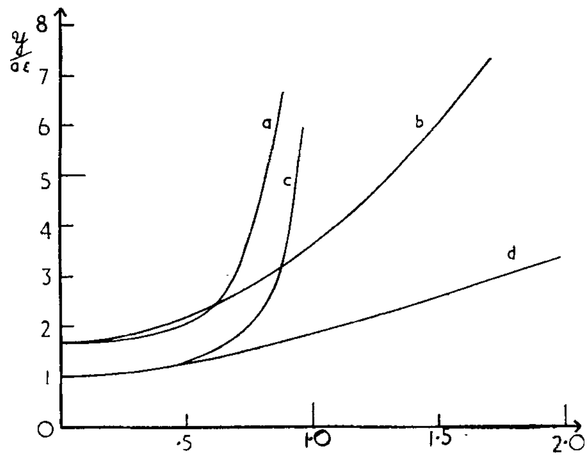

Figure 3. The distance $\mathscr{Y}$ of the shock wave away from the body as a function of the distance x from the stagnation point for (*a*) a circular cylinder ($\gamma = 1{\cdot}1$), (*b*) a parabolic cylinder ($\gamma = 1{\cdot}1$), (*c*) a sphere, and (*d*) a paraboloid of revolution. a is the radius of curvature of the body at the stagnation point.

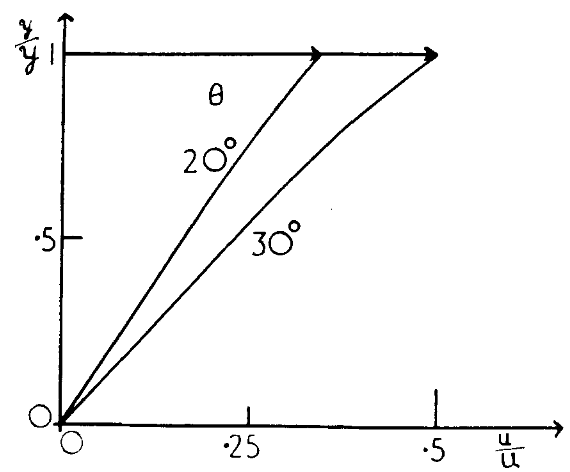

Figure 4. The first approximation to the velocity profiles in the boundary layer on a sphere at $\theta = 20°$ and $30°$.

5. Behaviour of solution near the stagnation point

As has been pointed out in the previous section, the formulae given there may not be valid near the stagnation point where both components of velocity are small. That these formulae do in fact remain true can be shown in the following manner. Near the stagnation point it is possible to consider the flow as incompressible, since the velocity components will all be small there. The simplification in the previous theory occurs because we are able to approximate the velocity term in Bernoulli's equation by including only one component of velocity. It is now no longer possible to do this. Nevertheless it will be seen that a somewhat fortuitous piece of cancelling enables us to overcome this difficulty.

Consider first the two-dimensional problem. The momentum equation in the normal direction is

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{u_\theta^2}{r} = 0, \tag{5.1}$$

where (r, θ) are polar coordinates with origin at the centre of curvature of the body near the stagnation point, and ρ is assumed constant. In the previous work we neglected the two 'convective' terms in the equation. Now, however, the velocity component u_r is no longer small, and thus we approximate the equation as

$$u_r \frac{\partial u_r}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{u_\theta^2}{r} = 0, \tag{5.2}$$

or, introducing the stream function ψ,

$$\frac{\partial}{\partial \psi}(p + \tfrac{1}{2}\rho u_r^2) - \frac{u_\theta}{r} = 0, \tag{5.3}$$

where ψ is chosen so that

$$\frac{\partial}{\partial r} = \rho u_\theta \frac{\partial}{\partial \psi}.$$

It will be noticed that the pressure in equation (3.8) has now been replaced by $p + \frac{1}{2}\rho u_r^2$. Consequently, if we consider Bernoulli's equation for incompressible flow, viz.

$$p + \tfrac{1}{2}\rho(u_r^2 + u_\theta^2) = \tfrac{1}{2}\rho q_*^2 + p_*, \tag{5.4}$$

where p_* and q_* are values on the shock wave, we see that replacing the pressure by $p + \frac{1}{2}\rho u_r^2$ reduces the problem once more to one containing the single component of velocity u_θ. Thus, integrating equation (5.3), we have

$$p + \tfrac{1}{2}\rho u_r^2 - \int_{\psi_s}^{\psi} \frac{u_\theta}{r}\, d\psi = p_s + \tfrac{1}{2}\rho u_{rs}^2, \tag{5.5}$$

where the suffix s denotes values on the shock wave at a station θ. Substituting for $p + \frac{1}{2}\rho u_r^2$ from equation (5.4) into equation (5.5) and differentiating with respect to ψ, we obtain

$$\rho u_\theta^2 \frac{\partial u_\theta}{\partial \psi} = \frac{\partial}{\partial \psi}(\tfrac{1}{2}\rho q_*^2 + p_*) - \frac{u_\theta}{r}. \tag{5.6}$$

Hence, by the cancellation of the term $\frac{1}{2}\rho u_r^2$, the problem is reduced once more to one containing u_θ only. Approximating the shock wave in this neighbourhood by $X = -c+(Y^2/2d)+O(Y^4)$, where (X, Y) are Cartesian coordinates with origin at the stagnation point and the X-axis along the line of symmetry, we then have

$$q_*^2 = \epsilon^2 U^2 + (U^2 Y^2/d^2) + O(Y^3),$$

and

$$p_* = (1-\epsilon)\rho_0 U^2[1-(Y^2/d^2)] + O(Y^4). \tag{5.7}$$

On the shock wave, $\psi \sim \rho_0 UY$; and hence, retaining only first order in ψ in (5.6), we obtain

$$u_\theta \frac{\partial u_\theta}{\partial \psi} = \alpha\psi - \beta u_\theta, \tag{5.8}$$

where

$$\alpha = \frac{1}{\rho_0^2 d^2}[1-2\epsilon+o(\epsilon)],$$

$$\beta = \frac{\epsilon}{\alpha\rho_0}[1+o(1)],$$

and a is the radius of curvature of the body at the stagnation point. Solving (5.8), we then obtain

$$\frac{\psi^2}{\psi_s^2} = \frac{\alpha-\beta u_{\theta s}-u_{\theta s}^2}{\alpha-\beta u_\theta - u_\theta^2}, \tag{5.9}$$

neglecting terms which are $o(1)$ on the right-hand side. Approximating (5.9) further, we have

$$u_\theta^2 - \frac{\psi^2}{\rho_0 a^2} = 3\epsilon U^2 \sin^2\theta, \tag{5.10}$$

assuming that

$$d = a[1+o(1)].$$

Also, since

$$(r-a)/a = \int_0^\psi [\rho u_\theta]^{-1}\, d\psi,$$

then

$$\frac{r-a}{a} = \tfrac{1}{2}\epsilon \log\left(\frac{4\psi^2}{3\epsilon\rho_0^2 U^2 a^2 \sin^2\theta}\right), \tag{5.11}$$

which is the limiting form of (4.23).

If we proceed in a similar manner for the three-dimensional case, we confirm the result (4.24) in the region of the stagnation point, but with an error term which is $O(\epsilon^{3/2})$ and not $O(\epsilon^2 \log^2\epsilon)$ as result (4.24) indicates.

6. The second approximation to the pressure

In §4, we obtained a second approximation to the shock shape and streamlines. From this result we can hence obtain the values of the variables in the region behind the shock wave to the second approximation.

To do this, we must consider a higher approximation to the equations of motion (3.5). The momentum equation in the y-direction taken from (3.1) and written in terms of the variables (x, ψ) becomes

$$\frac{u}{h}\frac{\partial v}{\partial x} - u^2\kappa + uk\frac{\partial p}{\partial \psi} = 0,$$

or

$$\frac{\partial p}{\partial \psi} = \frac{u\kappa}{k} - \frac{1}{hk}\frac{\partial v}{\partial x}. \tag{6.1}$$

In § 3, the second term on the right-hand side was neglected completely. We must now seek a higher approximation to the first term and a first approximation to the second term. Integrating (6.1), we obtain

$$p = p_s(x) + \int_{\psi_s}^{\psi}\left(\frac{u\kappa}{k} - \frac{1}{hk}\frac{\partial v}{\partial x}\right)d\psi. \tag{6.2}$$

From § 4, we know the equation of the streamlines,

$$y = f(x, \xi), \qquad \text{or } F(x, \psi), \tag{6.3}$$

say. Then the equation of the shock is given by

$$y = f(x, x) = \mathcal{Y}(x). \tag{6.4}$$

Thus the angle the shock makes to the oncoming flow is $\{\Phi + \tan^{-1}(\mathcal{Y}'(x)/h)\}$, whence the pressure on the shock is

$$\begin{aligned} p &= \rho_0 U^2(1-\epsilon)\sin^2\{\Phi + \tan^{-1}(\mathcal{Y}'(x)/h)\} \\ &= \rho_0 U^2\left\{(1-\epsilon)\sin^2\Phi + \sin 2\Phi\,\frac{\mathcal{Y}'(x)}{h_0(x)}\right\} \end{aligned} \tag{6.5}$$

to the next approximation.

From Bernoulli's equation, we have

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho} + \tfrac{1}{2}(u^2+v^2) = \tfrac{1}{2}U^2,$$

and hence

$$u^2 = U^2 - \left(\frac{1+\epsilon}{\epsilon}\right)\frac{p}{\rho} + o(\epsilon); \tag{6.6}$$

also, from the energy equation,

$$\frac{p}{\rho^\gamma} = \frac{p_*}{\rho_*^\gamma},$$

where p_*, ρ_* denote values at the shock wave. Thus,

$$\begin{aligned} u^2 &= U^2 - \frac{p_*}{\rho_0}\left(1+\epsilon+2\epsilon\log\frac{p}{p_*}\right) + o(\epsilon) \\ &= U^2\left\{\cos^2\Phi(x_s) - 2\epsilon\sin^2\Phi(x_s)\log\left(\frac{p(x, x_s)}{p_*(x_s)}\right) - \sin 2\Phi(x_s)\,\frac{\mathcal{Y}'(x_s)}{h_0(x_s)} + o(\epsilon)\right\}, \end{aligned} \tag{6.7}$$

where $x = x_s$ is the coordinate of the point on the shock where the streamline crosses it, and is related to the stream function by

$$\begin{aligned} \psi &= \rho_0 U[\eta(x_s) + \mathcal{Y}(x_s)\cos\Phi(x_s)] \\ &= \tfrac{1}{2}\rho_0 U[\eta(x_s) + \mathcal{Y}(x_s)\cos\Phi(x_s)]^2 \end{aligned} \tag{6.8}$$

in two and three dimensions respectively. Equation (6.7) then gives the second approximation to the velocity u.

From equation (6.3), it follows that

$$\frac{\partial\psi}{\partial x} = -\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial\psi},$$

or

$$v = \frac{1}{\rho hk}\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial\psi}. \tag{6.9}$$

Since we only require a first approximation to v, we can replace ρhk in (6.9) by its first approximation, and hence obtain

$$v = \frac{\epsilon}{h_0 k_0 \rho_0}\frac{P_*}{P}\frac{\partial F}{\partial x}\bigg/\frac{\partial F}{\partial\psi} + o\,(U\epsilon), \tag{6.10}$$

where P denotes the first approximation to the pressure given by (4.15), and P_* its value on the shock wave where the streamline ψ crosses it.

Let us now introduce a variable ξ to replace the stream function ψ, putting

$$\begin{aligned}\psi &= \rho_0 U\eta(\xi)\\ &= \tfrac{1}{2}\rho_0 U\eta^2(\xi)\end{aligned} \tag{6.11}$$

in two and three dimensions respectively, and let us write this $\psi = \rho_0 U\Psi(\xi)$ as before. The equation (6.10) becomes

$$v = \frac{\epsilon U}{h_0 k_0}\frac{P*}{P}\left(\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial\xi}\right)\frac{d\Psi}{d\xi} + o\,(U\epsilon). \tag{6.12}$$

The variable x_s is then related to ξ by

$$\eta(\xi) = \eta(x_s) + \mathcal{Y}(x_s)\cos\Phi(x_s),$$

or

$$x_s = \xi - \frac{\mathcal{Y}(\xi)\cos\Phi(\xi)}{\eta'(\xi)} + o\,(\epsilon); \tag{6.13}$$

and equation (6.7) can be written

$$u = U\cos\Phi(\xi)\left\{1 + \mathcal{Y}(\xi)\Phi'(\xi) - \frac{\mathcal{Y}'(\xi)}{h_0(\xi)}\tan\Phi(\xi) - \right.$$
$$\left. - \epsilon\tan^2\Phi(\xi)\log\left(\frac{P}{P_*}\right) + o\,(\epsilon)\right\}. \tag{6.14}$$

Noting in equation (6.2) that ψ_s corresponds to $\xi = x + [\mathcal{Y}(x)\cos\Phi(x)/\eta'(x)]$, we then obtain the pressure to the second approximation as

$$\frac{p}{\rho_0 U^2} = \sin^2\Phi - \epsilon\sin^2\Phi + \frac{\mathcal{Y}'}{h_0}\sin 2\Phi - \frac{\kappa_0}{k_0}\frac{\mathcal{Y}}{\eta'}\frac{d\Psi}{dx}\cos^2\Phi +$$
$$+ \int_x^\xi \left[\frac{\kappa(x,\zeta)}{k(x,\zeta)}\cos\Phi(\zeta)\left\{1 + \mathcal{Y}(\zeta)\Phi'(\zeta) - \frac{\mathcal{Y}'(\zeta)\tan\Phi(\zeta)}{h_0(\zeta)} - \right.\right.$$
$$\left.\left. - \epsilon\tan^2\Phi(\zeta)\log\left(\frac{P(x,\zeta)}{P_*(\zeta)}\right)\right\} - \frac{\epsilon}{h_0 k_0}\frac{\partial}{\partial x}\left\{\frac{P(x,\zeta)}{P_*(\zeta)}\frac{1}{h_0 k_0}\left(\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial\zeta}\right)\frac{d\Psi}{d\zeta}\right\}\right]\frac{d\Phi}{d\zeta}\,d\zeta, \tag{6.15}$$

where $\kappa(x,\zeta) = \kappa(x, y = f(x,\zeta)) = \kappa_0(x)[1-\kappa_0 f(x,\zeta)]$ to this approximation, $k \equiv 1$ or $\eta(x)+f(x,\zeta)\cos\Phi$ and

$$\frac{P}{P_*} = \frac{\sin^2\Phi - (\Phi'/k_0)\int_x^\zeta \cos\Phi(s)(d\Phi/ds)\,ds}{\sin^2\Phi(\zeta)}.$$

The arguments of the functions in (6.15) have been left out when they are simply x. The error in (6.15) is $O(\epsilon^2\log^2\epsilon)$ for two-dimensional flow, and $O(\epsilon^{3/2})$ for three-dimensional.

The expression (6.15) can be evaluated for any body shape when the function $\Phi(x)$ is known. In the case of a sphere, $\Phi = \frac{1}{2}\pi - \theta$, and the pressure on the body surface becomes

$$\frac{p}{\rho_0 U^2} = p_1 + \epsilon p_2, \tag{6.16}$$

where

$$p_1 = \frac{\sin 3\theta}{3\sin\theta},$$

and

$$p_2 = \cos^2\theta\{2\Lambda'\tan\theta - 1\} + \Lambda(\theta)\cos 2\theta - 2(1-\tfrac{1}{3}\sin^2\theta) - \\ - \frac{1}{\sin\theta}\int_0^\theta \Lambda(\phi)\cos 3\phi\,d\phi + \frac{1}{\sin\theta}\int_0^\theta \cos^3\phi \log\left[\frac{\sin 3\theta + \sin^3\phi}{3\sin\theta\cos^2\phi}\right]d\phi + \\ + \frac{9}{\sin\theta}\left\{\sin 3\theta\int_0^\theta \frac{\sin^3\psi(\sin^3\theta - \sin^3\psi)\,d\psi}{(\sin 3\theta + \sin^3\psi)^2} + \right. \\ \left. + 2\cos^2 3\theta\int_0^\theta \frac{\cos^3\psi(\sin^3\theta - \sin^3\psi)\,d\psi}{(\sin 3\theta + \sin^3\psi)^3}\right\},$$

with

$$\Lambda(\theta) = \int_0^\theta \frac{\cos^3\phi\,d\phi}{\sin 3\theta + \sin^3\phi}.$$

The final three integrals in p_2 and the function Λ can each be obtained in terms of simple functions, but lack of space prevents us from quoting them in

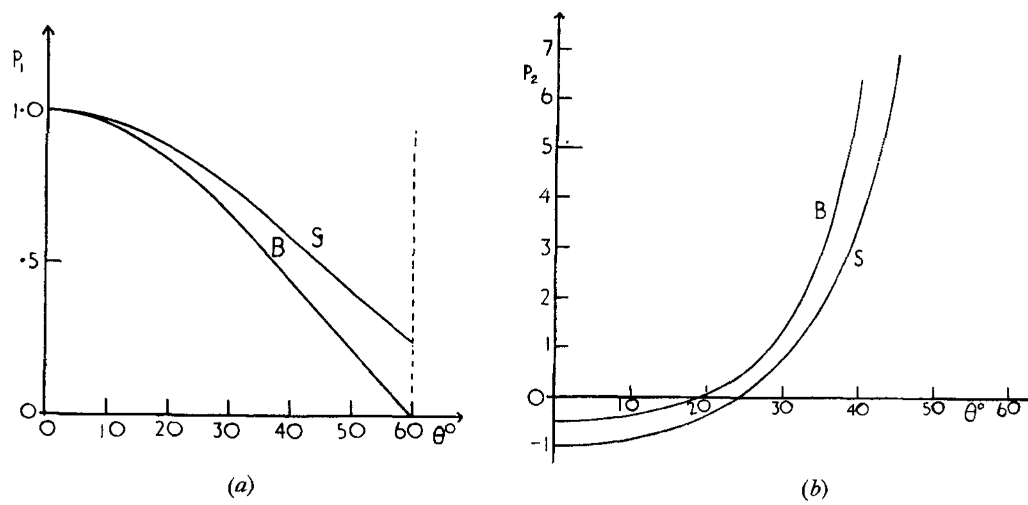

Figure 5. The pressure coefficients p_1 and p_2 on the body surface, and the shock wave for a sphere. Total pressure $p = \rho_0 U^2(p_1+\epsilon p_2)$. *S*, shock; *B*, body.

full. The functions p_1 and p_2 are plotted in figure 5. We see that near the singular point where p_1 is zero the pressure coefficient p_2 has a non-integrable singularity, i.e. $p_2 \sim (\sin 3\theta)^{-8/3}$ as $\theta \to \frac{1}{3}\pi$. This formulae can therefore not be expected to hold above $\theta \sim 35°$. It is also possible, if a little care is exercised in dealing with the stream function, to obtain from (6.16) a simple expression for the pressure along the axis of symmetry, which is

$$\frac{p}{\rho_0 U^2} = 1 - \frac{\epsilon}{2}\left[1 + \left(\frac{r-a}{a\epsilon}\right)^4\right], \tag{6.17}$$

giving a check on the result.

7. Gases with arbitrary thermodynamical properties

Under the conditions in which the previous theory holds, it becomes impossible to treat the gases as perfect with constant specific heats. It is necessary therefore to generalize the preceding theory to include gases which have quite arbitrary thermodynamical properties. In fact, it will be found possible to deduce the previous results for such a gas provided that we have a knowledge of how the enthalpy varies with pressure and density.

The shock equations can be written in the form given in equations (2.4). Provided, therefore, we know $i = i(p, \rho)$, these equations can be solved to give u_n, u_t, p and ρ on the shock wave. For K large, the magnitude of these variables is obtained from (2.4), and we can therefore assume as before that the shock wave rests on the body to a first approximation. Hence we obtain u and v on the shock wave. Behind the shock wave the variables will have magnitudes of this order and, in particular, $p/i\rho$ is small. We make the same approximations as before in the 'boundary layer' between shock and body. The final equation of motion (3.1), the energy equation, written in terms of the enthalpy and in the coordinates x and ψ, is

$$\frac{\partial i}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{7.1}$$

the derivatives being taken along streamlines. Since $p/i\rho$ is small behind the shock wave, equation (7.1) becomes, to the first approximation,

$$\frac{\partial i}{\partial x} = 0, \tag{7.2}$$

or $i = I(\xi)$, say.

Again, the velocity component normal to the body is small, and Bernoulli's equation states

$$i + \tfrac{1}{2}u^2 = \tfrac{1}{2}U^2. \tag{7.3}$$

Thus u is constant along streamlines. The momentum equation in the y-direction remains unchanged, and hence the pressure is given by (4.15) as before. Thus we know $p = p(x, \xi)$ and $i = I(\xi)$ from the shock conditions and (4.5). The enthalpy of the gas being given as a function of pressure and density, we thus have the density behind the shock as $\rho = \Omega[p(x,\xi), I(\xi)]$, say. Approximating the stream function as before in (4.9), and using the integral (4.17), we obtain the equation of the streamlines. In the two-dimensional case, however, it is again necessary to obtain a better approximation for u near the body, i.e. for ξ small. This is obtained from (7.1) by not completely neglecting the second term on the left-hand side, but replacing it by its first approximation.

Then,

$$i = I(\xi)+\int_{\zeta}^{x}\frac{1}{\rho(x,\zeta)}\frac{\partial p}{\partial x}\,d\zeta, \tag{7.4}$$

where $\rho(x,\zeta) = \Omega[p(x,\zeta), I(\zeta)]$, and p is given by (4.15). Equation (7.3) then gives the required approximation to u.

The formulae replacing (4.23) and (4.24) can then be written

$$y = \int_0^{\zeta}\frac{\sin^3\Phi(t)\,dt}{\left[\dfrac{\rho(x,t)}{\rho_0}\right]\left\{\cos^2\Phi(t)-2\displaystyle\int_0^{x}\frac{\rho_0}{\rho(x,s)}\frac{\partial R}{\partial x}\,ds\right\}^{1/2}}, \tag{7.5}$$

where $\rho(x,s) = \Omega(\rho_0 U^2 R(x,s), I(s))$, with $R(x,s)$ as defined in (4.23), for the two-dimensional case, and

$$y = \int_0^{\zeta}\frac{\eta(t)\sin^3\Phi(t)\,dt}{\rho(x,t)\cos\Phi(t)}, \tag{7.6}$$

where $\rho(x,t) = \Omega(\rho_0 U^2 Q(x,t), I(t))$, with $Q(x,t)$ as defined in (4.24) for the three-dimensional case. The formulae (7.6) and (7.7) then give the equations of the streamlines ξ = constant. On putting $\xi = x$, they give the equation of the shock wave.

Having obtained the shock shape $y = \mathcal{Y}(x)$, it is then not very difficult to generalize the expression for the pressure (6.15) in a like manner to obtain

$$\begin{aligned}\frac{p}{\rho_0 U^2} = {} & \sin^2\Phi(x)+H(\Phi(x))+\frac{\mathcal{Y}'(x)}{h_0(x)}\sin 2\Phi(x)-\frac{\kappa_0(x)}{k_0(x)}\frac{\mathcal{Y}(x)}{\eta'(x)}\frac{d\Psi}{dx}\cos^2\Phi(x)+\\ & +\int_x^{\xi}\left[\frac{\kappa}{k}\cos\Phi(\zeta)\left\{1+\mathcal{Y}(\zeta)\Phi'(\zeta)-\frac{\mathcal{Y}'(\zeta)}{h_0(\zeta)}\tan\Phi(\zeta)-\right.\right.\\ & \left.-\frac{\sec^2\Phi(\zeta)}{U^2}\int_{\zeta}^{x}\frac{1}{\rho(x,t)}\frac{\partial P}{\partial x}\,dt\right\}-\\ & \left.-\frac{1}{h_0 k_0}\frac{\partial}{\partial x}\left\{\frac{\rho_0}{\rho(x,\zeta)h_0 k_0}\left(\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial \zeta}\right)\frac{d\Psi}{d\zeta}\right\}\right]\frac{d\Psi}{d\zeta}\,d\zeta,\end{aligned} \tag{7.7}$$

where $y = f(x,\xi)$ is the equation of the streamlines from (7.5) or (7.6), $\rho(x,t) = \Omega(P(x,t), I(t))$ with P defined in (4.15), and

$$\kappa(x,\zeta) = \kappa(x, y = f(x,\zeta)),$$

etc. as in (6.15). The pressure on a shock wave of angle Θ has been written

$$p = \rho_0 U^2[\sin^2\Theta+H(\Theta)] \tag{7.8}$$

to this approximation, being deduced from the shock equations (2.4).

8. Conclusion

The theory developed above would seem to be useful in predicting the flow over many bluff bodies, both plane and axially symmetric. On all bodies, it gives a solution in regions near the nose for flows in which the Mach number is large and the density ratio across the shock wave large. However, difficulties arise in regions where the pressure falls below a certain value of the order of $\rho_0 U^2/K$. In these regions the solution is no longer valid. To obtain a solution at points beyond such a region, it would be necessary to make further assumptions about the flow.

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